$$z_0 = 2, \quad z = 2\sqrt{1 - \frac{(n-1)\,gx}{c_0^2}} \approx 2 - \frac{(n-1)\,gx}{c_0^2} - \left[\frac{(n-1)\,gx}{2c_0^2}\right]^2$$

and assuming that $\omega \gg (n-1)\,g\,/\,(4c_0)$, we find

$$u \approx \left[1 + \frac{(n-1)\,gx}{4c_0^2}\right] \int\limits_{-\infty}^{\infty} (u_1(\omega) \exp J_-^\circ + u_2(\omega) \exp J_+^\circ)\, d\omega$$

$$J_\pm^\circ = j\omega t \mp j\frac{\omega x}{c_0}\left(1 + \frac{(n-1)\,gx}{4c_0^2}\right)$$

4. An experiment was carried out with the view to determining the difference in the amplitude of signals produced by waves propagating up- and downward in a medium. Short pulse signals (10 - 20 and 60 - 80 nsec) were simultaneously applied to the two two-meter long aluminum rods fixed in the same manner and insulated by brass tubes. Pulses of an amplitude of 1 - 2 V were supplied to piezoelectric transducers attached to rod ends and completely insulated. Signals were fed to the bottom of one rod and to the top of the other. For maximum attenuation of wave reflection the rod ends were damped by rubber. The amplitude of the output signal was of the order of 0.5 - 1.5 mV.

This experiment had shown that when the acoustic wave propagates upward, the amplitude of output voltage was 1.2 - 1.5 mV, while in the case of wave propagating downward this amplitude was 0.4 - 0.5 mV. The results of derived solutions were thus qualitatively confirmed. It is interesting to note that the accuracy of this experiment was sufficient for demonstrating the investigated phenomenon in spite of the small length of the rod.

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APPROXIMATE EQUATIONS FOR WAVES IN MEDIA WITH SMALL NONLINEARITY AND DISPERSION

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The method of simplifying systems of equations with small nonlinearities and dispersion is considered. Such systems differ from the linear hyperbolic system by a certain integro-differential operator with a small parameter. Method is based on the reduction of input equations to the normal form and subsequent recurrent procedure. In the case of a wave propagating along one of the characteristics of the system (single-wave processes) the first approximation by this method leads to known Burgers, Korteweg-de Vries, Klein-Gordon, and others equations which were first derived for specific physical models, and later for a more general system of

nonlinear equations [1, 2]. However the proposed method can be applied to multi-wave interaction, when waves corresponding to two or more characteristics of the system are taken into consideration. It is also shown that in the case of single-, as well as multi-wave processes the order of simplified equations increases with the order of approximation, and the increase is different for systems with high and low dispersion frequencies (when the small parameter is contained in terms with higher derivatives or those containing integrals of the unknown function, respectively).

Let us consider a system of the form

$$\partial u / \partial t + B(\tau, \rho) \nabla u = \varepsilon F\{u\} \tag{1}$$
$$(\tau = \tau_0 + \varepsilon t,\ \rho = \rho_0 + \varepsilon \mathbf{r},\ \varepsilon \ll 1)$$

where u is the N-dimensional vector of field variables, $\mathbf{B} = \{B_x, B_y, B_z\}$ is the set of three square matrices, F are specified, generally nonlinear integro-differential operators in terms of $\mathbf{r}$ and t which depend on ε, τ and ρ. We assume that all eigenvalues of matrix B are real, i.e. that for $\varepsilon = 0$ system (1) is hyperbolic (it will become clear subsequently that this condition need not necessarily be satisfied, since only the existence of $r < N$ real characterisitcs of system (1) is necessary when $\varepsilon = 0$).

For $\varepsilon = 0$ the solution of system (1) is of the form of plane waves

$$u = U_0 + \psi U\,(t - \mathbf{V r} / V^2) \tag{2}$$

where U_0 and U are arbitrary constant vector and scalar function, respectively, which are determined by boundary and initial conditions, and ψ is the right-hand eigenvector of matrix $\mathbf{B}$, which corresponds to the eigenvalue of $\mathbf{V}$

$$\mathbf{VB}\psi = V^2\psi, \qquad \mathrm{Det}\,|\,\mathbf{VB} - V^2\,| = 0 \tag{3}$$

To examine the more general class of quasi-plane waves propagating along the x-axis for $\varepsilon \neq 0$ we reduce the input system (1) to the normal form (cf. [3]) by substituting the variables

$$u = Yv \tag{4}$$

where Y is a square matrix composed of linearly independent eigenvectors of matrix $B \equiv B_x$. We obtain

$$\frac{\partial v}{\partial t} + \lambda(\tau, \rho)\frac{\partial v}{\partial x} + \varepsilon \mathbf{C}(\tau, \rho)\,\nabla_{\perp} v = \varepsilon Y^{-1}\left[F - \left(\frac{\partial Y}{\partial t} + \mathbf{B}\nabla Y\right)v\right] \equiv \varepsilon f \tag{5}$$
$$\left(\mathbf{C} = Y^{-1}\mathbf{B}_{\perp}Y,\ \ \mathbf{B}_{\perp} = \{B_y, B_z\},\ \ \nabla_{\perp} = \left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}\right)$$

where λ is a diagonal matrix with elements $V_1, \ldots V_N$, and Y^{-1} is the inverse of matrix Y.

The conversion to system (5) considerably simplifies the construction of the scheme of successive approximations with respect to parameter ε. The unknown function, in accordance with (4) and (5) is a superposition of "almost traveling" waves, each of which propagates along one of the characterisitcs. Boundary or initial conditions are in a number of cases such that for $\varepsilon = 0$ only $r < N$ waves $v_i\,(x - V_i t)$ or even only one wave are induced in the medium. For example, in the case of a specified wave from region $x < 0$ impinging on the boundary of a semi-infinite medium $(x > 0)$ all characteristics with $V < 0$ are excluded from the analysis in the first approximation. Another example is the problem with initial conditions in the form of localized perturbation which decomposes into pulses propagating along characteristics without overlapping, so that for reasonably long t each of these can be examined independently. It is possible to separate in similar cases r "principal" variables among components of vector v and calculate the remaining $N - r$ components, which are small, by using the theory of perturbations. As the result in each (m- th) approximation with respect to the small parameter ε Eqs. (5) are divided into two groups: r equations of the form

$$\frac{\partial v_i}{\partial t} + V_i(\tau, \rho)\frac{\partial v_i}{\partial x} + \varepsilon \sum_{j=1}^{r} \mathbf{C}_{ij}(\tau, \rho)\nabla_\perp v_j = \varepsilon f_i\{v_i, v_s^{(m-1)}, \varepsilon\} - \varepsilon \sum_{s=r+1}^{N} \mathbf{C}_{is}(\tau, \rho)\nabla_\perp v_j^{(m-1)} \quad (i = 1, \ldots, r) \tag{6}$$

and $N - r$ linear equations for $v_s^{(m)}$

$$\frac{\partial v_s^{(m)}}{\partial t} + V_s\frac{\partial v_s^{(m)}}{\partial x} = \varepsilon f_s\{v_i, v_s^{(m-1)}, \varepsilon\} - \varepsilon \sum_{i=1}^{r} \mathbf{C}_{si}\nabla_\perp v_i - \varepsilon \sum_{l=r+1}^{N} \mathbf{C}_{sl}\nabla_\perp v_l^{(m-1)} \tag{7}$$

It is evident that (7) asymptotically decomposes into $N - r$ independent first order equations with known right-hand parts; as the result the construction of the approximate system (6) reduces to iterations. We note that this scheme of derivation of simplified equations is simpler than that presented in [1, 2] even in the case of single wave problem.

Let us consider the structure of the right-hand side of (6). In the first approximation it is necessary to calculate function f_i appearing there for $\varepsilon = 0$, i.e. to set $v_s = 0$. For a given r the order of first approximation equations is obviously dependent on the form of functional F in the input system (1). In many important cases F can be presented in the form of a sum of derivatives of integrals of certain functions of the unknown variable. A similar presentation of f is obviously possible. The first approximation equations are then of the form

$$\frac{\partial v_i}{\partial t} + V_i\frac{d v_i}{\partial x} + \varepsilon \sum_{j=1}^{r} \mathbf{C}_{ij}\nabla_\perp v_j + \ldots + \varepsilon\int h_i^{(-1)}\,dx + \varepsilon h_i^{(0)} + \varepsilon\frac{\partial h_i^{(1)}}{\partial x} + \varepsilon\frac{\partial^2 h_i^{(2)}}{\partial x^2} + \varepsilon\frac{\partial^3 h_i^{(3)}}{\partial x^3} + \ldots = 0 \tag{8}$$

where $h^{(n)}$ is a function of v_i and all derivatives with respect to t are eliminated by using (6). In the simplest case only function $h^{(1)}$ is nonzero, which relates to a nondispersive medium. If in addition to $h^{(1)}$ function $h^{(2)}$ ("viscosity") or $h^{(3)}$ (high frequency dispersion) are nonzero and the latter are linear with respect to v and $h^{(1)} \sim v^2$, then for $r = 1$ the known Burgers and Korteweg-de Vries equations follow from (8). The $h^{(0)}$ term defines "low-frequency losses". Finally, when function $h^{(-1)}$ is nonzero, then differentiating (8) with respect to t and eliminating $\partial^2 v/\partial x \partial t$, we obtain for $r = 1$ the Klein-Gordon equation. In the nonlinear wave theory each of these equations defines in the single-wave approximation the simplest form of nonlinear, dispersion, and dissipative properties of a medium. It will be seen from (8) that similar equations can be written for any number of interacting waves.

In subsequent approximations, e.g. in the m-th one, function f is calculated by known v_s^{m-1} to within ε^{m-1}. Since V_s does not coincide with any of the V_i (owing to the hyperbolic properties of system (1)), V_s is determined as an integral of the right-hand part of Eqs. (7). The substitution of v_s into (6) shows that the order of system (6) increases, generally speaking, in each approximation, although the increase differs depending on the form of functional F.

Let initially $F \sim \partial v / \partial x$, then terms of the kind

$$\frac{\partial f}{\partial(\partial v / \partial x)} \frac{\partial v_s}{\partial x}$$

lead to the appearance in the second approximation of the term $\partial v_i / \partial x$ in (6). This also takes place in higher approximations. Hence for a nondispersing medium the order of system (6) does not increase.

If $F \sim \partial^2 v / \partial x^2$, then in every approximation the order of system (6) increases by r and because of

$$\frac{\partial f}{\partial(\partial^2 v / \partial x^2)} \frac{\partial^2 v_s}{\partial x^2}$$

terms $\partial^3 v / \partial x^3$ appear in the second approximation, and so on. Corrections of this kind to the Burgers equation in acoustics were obtained in [4]. The term with $\partial^3 v / \partial x^3$ increases the order by $2r$, and so on.

If, however, $F \sim v$, then we obtain in the second approximation $\int v dx$, and integrals of increasing multiplicity appear with increasing approximation order. This also relates to all cases, when F contains integrals of v (for $F \sim \int v dx$ the multiplicity of integrals increases by $2r$ and so on). Thus the increase of the order of approximate equations for systems with high- and low-frequency dispersion (dissipation) is not the same. While in the first case the order of derivatives increases, in the second it is the multiplicity of integrals. The "border-line" case is that of the nondispersing medium, when the small parameter at the first approximations in (1) and the order of system (6) do not vary in any approximation.

Finally, because of the presence of $C_{is} \nabla_{\perp} v_s$ a term of the kind $\Delta_{\perp} \int v dx$, which is related to the diffusion approximation for paraxial nonsinusoidal beams (first investigated in acoustics [5]), appears in (6).

We also note that power of nonlinear terms increases, as usual, with the order of approximation.

We note in conclusion that the transformation of the input system to approximate equations (6) often simplifies the choice of optimum methods of their solution. Thus, if in the first approximation the solution of (6) is in the form of a set of stationary waves, methods of averaging may be applied in the subsequent approximation [6, 7].

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CONTACT PROBLEM FOR A STAMP WITH NARROW RECTANGULAR BASE

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The problem of impression of a stamp with narrow rectangular base into an elastic isotropic half-space under the effect of a vertical force is considered. This problem has been studied in [1, 2]. Asymptotic properties of the integral equation obtained, which goes over into a singular integral in the limit as the beam width diminishes permitting substantiation of the known Zimmerman-Winkler hypothesis, were established in [1]. An approximate solution of the integral equation from [1] was given in [2]. A brief survey of the research devoted to the problem of impressing a rectangular stamp is contained in [2, 3]. A more complete method of solving this problem is proposed below.

1. Let us consider a stamp in the shape of a narrow rectangle of length $2a$ and width 2δ, where $\varepsilon = \delta / a \ll 1$. Let a vertical force P impress this stamp into an elastic isotropic half-space $z \geqslant 0$. The force P passes through the center of gravity of the stamp and is directed along the z-axis.

Applying a two-dimensional Fourier integral transform to the Lamé equilibrium equations in rectangular xyz coordinates, we find

$$w(x, y, 0) = -\frac{1-\nu^2}{\pi E} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha^2 + \beta^2)^{-1/2} \sigma_z^{**}(\alpha, \beta, 0) e^{-i(\alpha x + \beta y)} d\alpha \, d\beta \qquad (1.1)$$

Here E, ν are the Young's modulus and the Poisson's ratio of the material of the elastic half-space, respectively, w is the projection of the displacement vector on the z-axis, σ_z^{**} is the two-dimensional Fourier tranform of the normal stress σ_z. Formula (1.1) is valid under the condition of no shear stresses on the half-space boundary (at $z = 0$). This formula establishes the connection between vertical displacements of the half-space